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Self-similar viscous gravity currents: phase-plane formalism

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A theoretical model for the spreading of viscous gravity currents over a rigid horizontal surface is derived, based on a lubrication theory approximation. The complete family of self-similar solutions of the governing equations is investigated by means of a phase-plane formalism developed in analogy to that of gas dynamics. The currents are represented by integral curves in the plane of two phase variables, Z and V, which are related to the depth and the average horizontal velocity of the fluid. Each integral curve corresponds to a certain self-similar viscous gravity current satisfying a particular set of initial and/or boundary conditions, and is obtained by solving a first-order ordinary differential equation of the form $\mathrm{d}V/\mathrm{d}Z = f(Z, V)$, where f is a rational function. All conceivable self-similar currents can thus be obtained. A detailed analysis of the properties of the integral curves is presented, and asymptotic formulae describing the behaviour of the physical quantities near the singularities of the phase plane corresponding to sources, sinks, and current fronts are given. The derivation of self-similar solutions from the formalism is illustrated by several examples which include, in addition to the similarity flows studied by other authors, many other novel ones such as the extension to viscous flows of the classical problem of the breaking of a dam, the flows over plates with borders, as well as others. A self-similar solution of the second kind describing the axisymmetric collapse of a current towards the origin is obtained. The scaling laws for these flows are derived. Steady flows and progressive wave solutions are also studied and their connection to self-similar flows is discussed. The mathematical analogy between viscous gravity currents and other physical phenomena such as nonlinear heat conduction, nonlinear diffusion, and ground water motion is commented on.

1. Introduction

Viscous gravity currents occur in many situations of interest in geophysics, industrial engineering, geology, and environmental sciences (see Huppert 1986, for a geophysically oriented review, also Kerr & Lister 1987; Simpson 1982; and Hoult 1972). The main feature of these currents is that flow is primarily horizontal and is governed by a balance between gravity and viscous forces, inertia effects being negligible (creeping flow). We refer to the work of Huppert (1982) for a discussion of the conditions for which the viscous forces overwhelm inertial forces, so that the latter can be ignored. Usually, except perhaps at the beginning of the phenomenon, the length of the current greatly exceeds its thickness, and this justifies the use of the approximation of lubrication theory (Huppert 1982). The equations of this model admit a family of similarity solutions, which represent the asymptotics of a broad class of flows corresponding to a variety of initial and/or boundary value problems.

Self-similar viscous gravity currents have been studied theoretically by Huppert (1982). He considered plane and axisymmetric currents on a rigid horizontal surface produced by a source whose flux depends on time according to a power law and obtained the solutions by numerical integration of a second-order differential equation (only exceptional cases admit analytic solutions). Experiments have been carried out by Britter (1979), Didden & Maxworthy (1982), Huppert (1982), and Maxworthy (1983).

Similarity solutions describing other types of gravity currents have been studied in connection with the spread of oil on the surface of the ocean in a viscosity-dominated regime by Fay (1969) and by Hoult (1972) using a different approach. The similarity viscous spread of subducted lithospheric material along the mid-mantle boundary has been considered by Kerr & Lister (1987). Gravity currents in the inertia-dominated regime have been investigated by Grundy & Rottman (1985, 1986).

These flows are only a small part of the family of similarity solutions of the governing equations. Many other instances of self-similar viscous gravity currents can be conceived, as will be shown here. To mention just a few examples in this introduction, the flow produced in plane geometry by the removal of a wall that separates two pools of fluid of different depth is self-similar (as in the analogous classical problem of the breaking of a dam), also the flow produced by a source that discharges fluid into a pool having initially a uniform depth is self-similar in axial symmetry if the flux of the source is constant in time, and in plane geometry if the flux varies as the $-\frac{1}{2}$ power of time. Other instances of flows over plates of finite extent are also self-similar, and will be discussed in this paper.

The theoretical interest of a systematic investigation of the similarity solutions of the governing equations of viscous gravity currents is further enhanced by the fact that the mathematics involved is essentially equivalent to that of several other physical phenomena also governed by nonlinear parabolic equations (see Seshadri & Na 1985), such as nonlinear diffusion and nonlinear heat conduction (transport of heat by radiation in multiply or fully ionized gases, electron heat conduction in plasmas, etc.); the latter deals with strong thermal waves such as may occur in explosions, and in laser–plasma interaction. We can also quote the equations of the Dupuit–Forchheimer idealization for the flow of ground water, the so-called porous media equation (see for example Peletier 1981), and the theory of electric transmission in cables coated with resistive paints that exhibit nonlinear characteristics, as additional examples of models having a mathematical structure equivalent to that we are considering in this paper (Bear 1972; Boyer 1962; also Seshadri & Na 1985). Many results concerning viscous gravity currents can then be applied, *mutatis mutandis*, to these problems. In this connection it can be observed that the similarity solutions for nonlinear diffusion derived by Pattle (1959), and the self-similar solutions of the nonlinear heat conduction equation given by Barenblatt (1979), Tappert (1977) and Barenblatt & Zel'dovich (1972) are equivalent to some self-similar viscous gravity currents discussed by Huppert (1982). Among the vast literature concerning similarity solutions for nonlinear parabolic equations analogous to that considered here we can also mention the papers of Pert (1977), Grundy (1979) and Smyth & Hill (1988), among others.

The primary aim of this paper is to develop a comprehensive theory of viscous gravity currents on a rigid horizontal surface which allows one to obtain systematically the complete family of self-similar solutions of the governing equations of the model. To this end we use a phase-plane formalism analogous to that

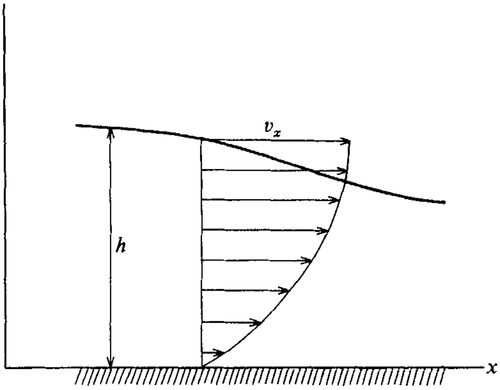

FIGURE 1. Geometry of the problem.

developed by Sedov (1959) and Courant & Friedrichs (1948) for gas dynamics (see also Zel'dovich & Raizer 1967). By means of this technique the problem is reduced to the integration of an autonomous first-order differential equation whose solutions are certain integral curves in the phase-plane. The integral curves represent the actual flow, whose description in terms of the physical variables requires an additional quadrature. Using this method, we derive many new self-similar solutions that describe currents corresponding to various boundary or initial conditions. The flows studied by Huppert (1982) are contained in our theory. The behaviour of the integral curves in the phase-plane is described and their asymptotic properties near the singularities is analysed. In addition we investigate steady currents and progressive waves, as these flows are closely related to the self-similar currents. A more detailed study of some solutions that represent especially interesting flows, and the analysis of their stability is left for a forthcoming paper.

2. Basic equations and formalism

The governing equations of slow viscous gravity flows on a rigid horizontal surface are obtained assuming that the motion is essentially horizontal, so the pressure is purely hydrostatic ($\partial p/\partial z = -\rho g$), inertia effects are negligible, and that the length of the current is much larger than its depth (Buckmaster 1977, see also Huppert 1982). For alternative derivations and extensions see Smith (1969) and Nakaya (1974). Figure 1 illustrates the geometry of the problem. Let x denote the horizontal coordinate (Cartesian in the case of unidirectional flow, or radial if the flow is axisymmetric), z the vertical coordinate and t the time. The acceleration of gravity will be denoted by g, and ν is the kinematic viscosity. If $h(x,t)$ denotes the thickness of the current, the boundary conditions of no slip at the bottom and no tangential stress at the upper free surface require that the horizontal velocity v_x depend on z as

$$v_x = \frac{3vz}{2h^2}(2h-z), \tag{1}$$

where $v(x,t) = \frac{2}{3}v_x(z=h)$ is the average horizontal velocity. One then obtains:

$$h^2\frac{\partial h}{\partial x}+v = 0, \tag{2}$$

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(vh)+n\frac{vh}{x} = 0. \tag{3}$$

Here $n = 0, 1$ according to whether the flow is unidirectional (plane symmetry) or axisymmetric. In (2) and (3) we have introduced a new dependent variable $\mathscr{h}$ defined as

$$\mathscr{h} = (g/3\nu)^{\frac{1}{3}}h, \tag{4}$$

to take advantage of the fact that the dimensional parameter g/ν appears in the problem only through this combination.

So far, nothing has been said about the boundaries of the horizontal supporting surface, or plate. This can be infinite, or may be limited by walls, or by borders over which the liquid may spill; the walls and the borders may be located at fixed positions or may be movable. These and other boundary or initial conditions will be considered as appropriate when discussing specific currents.

It can be observed that a viscous gravity current of a heavy fluid whose density is $\rho_2 = \rho$ that intrudes in a lighter ambient fluid whose density is $\rho_1 = \rho - \rho'$ is equivalent to the flow of a single fluid with a free upper surface, such as we are considering, provided one replaces g by $g' = g\rho'/\rho$ (see Huppert 1982). Then, the results of this paper may be carried over to this case, with the appropriate substitutions.

Using (2) to eliminate v in (3) one obtains a second-order differential equation for $\mathscr{h}$ of the general form

$$\frac{\partial \mathscr{h}}{\partial t} = x^{-n}\frac{\partial}{\partial x}\left(x^n \mathscr{h}^m \frac{\partial \mathscr{h}}{\partial x}\right), \tag{5}$$

where m is some number ($m = 3$ in the present case). An equivalent equation was used by Huppert (1982) as the starting point of his analysis. We shall follow a different procedure, and retain both dependent variables, v and $\mathscr{h}$, to introduce the phase-plane formalism.

We notice in passing that (5) can be regarded as a nonlinear heat conduction equation if one interprets h as the temperature ϑ and if the heat conductivity λ is a power function of ϑ of the form $\lambda = \lambda_0 \vartheta^m$, provided one identifies $(g/3\nu)^{\frac{1}{3}}$ with $(\lambda_o/c)^{1/m}$ (c = heat capacity). Here $m = \frac{13}{2}$ for transport of thermal energy by radiation in a completely ionized gas, $m = 4.5$–5.5 for multiply ionized gases, and $m = \frac{5}{2}$ for electron heat conduction in a plasma (Boyer 1962). A similar equivalence can be also established with respect to the equations of nonlinear diffusion, the Dupuit–Forchheimer equations [$m = 1$], and the nonlinear electric transmission. Other physical situations giving rise to similar equations are mentioned by Peletier (1981) and Lacey, Ockendon & Tayler (1982) where additional references can be found.

Since the governing equations (2), (3) do not contain any constant dimensional parameter and involve only quantities having the dimensions of length [L], time [T] or combinations of both, $\mathscr{h}$ and v can be expressed as

$$\mathscr{h} = (x^2 t^{-1} Z)^{\frac{1}{3}}, \quad v = xt^{-1}V, \tag{6}$$

where Z and V are, in general, dimensionless functions of x, t and the constant parameters of the problem, which arise from the initial conditions and the boundary conditions. Substituting in (2) and (3) one obtains

$$x\frac{\partial Z}{\partial x} + 2Z + 3V = 0, \tag{7}$$

$$t\frac{\partial Z}{\partial t} + 3xZ\frac{\partial V}{\partial x} + xV\frac{\partial Z}{\partial x} + (5+3n)\,VZ - Z = 0. \tag{8}$$

If the boundary or initial conditions involve two (or more) constant parameters with independent dimensions, it will be always possible to find two combinations ℓ and $\imath$ of them such that $[\ell] = L$, $[\imath] = T$. Then Z and V will in general depend on two dimensionless variables $x/\ell, t/\imath$ (and also on some dimensionless constant parameters $\pi_1, \pi_2, \ldots$ if there are more than two constant governing parameters), so that the problem will not be self-similar.

Let us assume now that the problem involves only one parameter b with independent dimensions. Clearly it can be assumed without loss of generality that

$$[b] = LT^{-\delta}, \tag{9}$$

where δ is a numerical constant. Then there will be a single dimensionless combination of x, t and b, which we can take as

$$\zeta = x/bt^{\delta}. \tag{10}$$

In this case the motion is self-similar, ζ being the similarity variable, and $Z = Z(\zeta)$, $V = V(\zeta)$.

For self-similar flows the phase variables Z and V satisfy the following ordinary differential equations:

$$\zeta Z' + 2Z + 3V = 0, \tag{11}$$

$$3\zeta Z V' - \zeta Z'(\delta - V) + (5 + 3n)\, VZ - Z = 0, \tag{12}$$

where the prime denotes the derivative with respect to ζ.

Eliminating ζ from (11) and (12) one obtains after a little algebra

$$\frac{\mathrm{d}V}{\mathrm{d}Z} = \frac{Z(2\delta - 1) + 3(1+n)\, VZ + 3(\delta - V)\, V}{3Z(2Z + 3V)}, \tag{13}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}Z}(\ln|\zeta|) = -\frac{1}{2Z + 3V}. \tag{14}$$

In the case of nonlinear heat conduction, an analogous derivation leads to:

$$\frac{\mathrm{d}V}{\mathrm{d}Z} = \frac{Z(2\delta - 1) + m(n+1)\, ZV + m(\delta - V)\, V}{mZ(2Z + mV)}, \quad \frac{\mathrm{d}}{\mathrm{d}Z}(\ln|\zeta|) = -\frac{1}{2Z + mV}, \tag{15}$$

in place of (13) and (14), where

$$\vartheta = (cx^2 Z/\lambda_0 t)^{1/m}, \quad \varphi = c\vartheta x V/t, \tag{16}$$

with φ denoting the heat flow; in this case $n = 0, 1, 2$ for plane, cylindrical and spherical geometry, respectively.

The solution of a self-similar problem is thus essentially reduced to the integration (numerical, in general) of the autonomous first-order ordinary differential equation (13). Once $V(Z)$ is known, (14) can be integrated to obtain $\zeta(Z)$; inversion then allows to obtain $Z(\zeta)$ and $V(\zeta)$.

The (Z, V)-plane will be called the 'phase plane', according to common usage. A solution of (13) is represented by a curve in the phase plane, which is called an 'integral curve'. A single integral curve passes through any regular point of the phase plane. Any integral curve represents a self-similar flow of a certain sort. The solution of a given self-similar problem characterized by some particular boundary conditions is represented in the phase plane by one or more pieces of the appropriate integral curve (or curves) and must satisfy at its ends the boundary conditions. Each piece represents the flow in a certain domain of the independent variables. If the required solution is represented by more than a single piece, the flows corresponding to the individual pieces must be adequately matched at the common boundary of their respective domains, as will be shown when discussing the examples. In order to determine which integral curve corresponds to the problem at hand (i.e. to the given initial and boundary conditions) it is necessary to know the behaviour of the solutions in the neighbourhood of the singular points of (13). The whole (Z, V)-plane needs to be considered, and according to (6) solutions having $Z > 0$ correspond to $t > 0$, while solutions for which $Z < 0$ are meaningful for $t < 0$.

3. Investigation of the integral lines in the phase plane

The relevant results of this section are summarized in tables 1 and 2 for easy reference.

Let us discuss in detail the properties of the integral curves near the singular points of (13). In this Section we shall denote by K and K' the integration constants that arise from (13) and (14), respectively, or from their approximations near the singularities; we shall also omit the absolute value bars around ζ as no confusion can arise. Six singular points of (13) can be recognized and will be examined in turn.

(i) Point O $(Z_O = 0, V_O = 0)$. Except for the special case $\delta = 0$, which will be considered separately, O represents points at infinity of the fluid ($\zeta = \infty$). The behaviour of the integral curves near O is different in the half-planes $Z > 0$ and $Z < 0$. For $Z > 0$, O is a node: as O is approached all integral curves (except $Z = 0$, which is of no interest) converge to a curve given by

$$V = -\frac{2\delta-1}{3\delta}Z\left[1-\frac{(5+3n)\,\delta-4}{3\delta^2}Z+\dots\right]. \tag{17}$$

For $Z < 0$, O is a saddle: only a single curve, given near O by (17) arrives at O. In either case the following asymptotic formulae hold $[\delta \neq 0, \frac{1}{2}]$ for the curves that reach O:

$$Z = K'\zeta^{-1/\delta}, \tag{18}$$

$$h = (K'\nu/g)^{1/3}(b/3)^{1/3\delta}x^{(2\delta-1)/3\delta}, \tag{19}$$

$$v = -K'\frac{2\delta-1}{3\delta}b^{1/\delta}x^{(\delta-1)/\delta}. \tag{20}$$

We notice that h and v do not depend on t as $x \to \infty$.

When $\delta = \frac{1}{2}$ and $Z > 0$ the integral curves are given near O by

$$V = K\exp(-1/4Z), \tag{21}$$

and one has the asymptotic formulae:

$$Z = K'\zeta^{-2}, \quad h = (3b^2K'\nu/g)^{\frac{1}{3}}, \quad v = (Kx/t)\exp(-\zeta^2/4K'). \tag{22}$$

For $Z < 0$ only the curve corresponding to the trivial solution $V = 0$ arrives at O.

Finally, in the case $\delta = 0$, the point O is a saddle. Near O one has, in addition to the trivial solution $Z = 0$,

$$V = \pm(\tfrac{1}{9}KZ^{-\frac{2}{3}} - \tfrac{2}{15}Z)^{\frac{1}{2}}. \tag{23}$$

Only the curve corresponding to $K = 0$ in (23), given by

$$V = \pm(-2Z/15)^{\frac{1}{2}}, \tag{24}$$

passes through O. Moving along this curve, O represents a fixed front (we call 'front' a place where the thickness h of the fluid vanishes), located at finite distance from the origin of the coordinates. For $n = 0$ (plane symmetry) (24) is an exact analytic solution of (13). Its properties will be discussed later on.

(ii) Point A $[Z_A = 0, V_A = \delta]$ is a saddle. If $\delta = 0$, A coalesces with O, then only $\delta \neq 0$ needs to be considered. Two integral curves pass through A. One is $Z = 0$, and the other, which we denote by $\mathscr{A}$, is given approximately by

$$V = \delta + \frac{(5+3n)\,\delta - 1}{12\delta} Z. \tag{25}$$

The point A represents the advancing front of a gravity viscous current. We denote the front coordinate by x_f $(= \zeta_f b t^{\delta}, \zeta_f = \text{const.})$ and introduce the notation $\eta = x/x_f = \zeta/\zeta_f$. Then, near x_f,

$$Z = 3\delta(1-\eta), \tag{26}$$

and

$$h = \zeta_f^{\frac{2}{3}} \left(\frac{9b^2\nu\delta}{g}\right)^{\frac{1}{3}} t^{(2\delta-1)/3} (1-\eta)^{\frac{1}{3}} \left[1 - \frac{(1+3n)\,\delta - 1}{24\delta}(1-\eta) + \ldots\right], \tag{27}$$

$$v = \delta x_f t^{-1} \eta \left[1 + \frac{(5+3n)\,\delta - 1}{4\delta}(1-\eta) + \ldots\right]. \tag{28}$$

For $Z > 0$, the curves $\mathscr{A}$ represent the currents produced by sources whose flux depends on time according to power laws (see Huppert 1982, and Didden & Maxworthy 1982). Analytic formulae for curves of type $\mathscr{A}$ can be obtained for $n = 0, \delta = \frac{1}{5}$, when $\mathscr{A}$ is given by $V = \frac{1}{5}$, and for $n = 1, \delta = \frac{1}{8}$, when the curve is $V = \frac{1}{8}$; these correspond to the plane and axisymmetric spreading of a fixed volume of fluid, and yield the solutions first obtained by Pattle (1959) and later by Smith (1969), Nakaya (1974) and again by Lopez, Miller & Ruckenstein (1976). In addition, there are also exact analytic solutions of (13) for $n = 0$, in the cases $\delta = 1$ and $\delta = \frac{1}{8}$. In the first case $\mathscr{A}$ is given by

$$Z = 3V(V-1). \tag{29}$$

This curve represents a current whose profile moves with constant velocity, without changing its shape. For the case $n = 0, \delta = \frac{1}{8}$ the trajectory given by (25) is an exact solution for any Z and represents a current that is drained from the origin and that has a front moving away. It is analogous to the so-called dipole-type solutions first obtained by Barenblatt (1954) and Barenblatt & Zel'dovich (1957) (see Zel'dovich & Raizer 1967) in the context of nonlinear heat conduction and the porous media equation. Both cases will be discussed in detail later on.

(iii) Point B $(Z_B = -3/2(5+3n), V_B = 1/(5+3n))$ is a node for $\delta \leqslant \delta_-$ and for $\delta \geqslant \delta_+$, where $\delta_{\pm} = 13/10 \pm (6/5)^{\frac{1}{2}}$ if $n = 0$, and $\delta_{\pm} = 1 \pm \sqrt{3}/2$ if $n = 1$. With the notation $Z^* = Z - Z_B$, $V^* = V - V_B$, the approximate formula for the integral curves in the neighbourhood of B is

$$(V^* - \gamma_+ Z^*)^{\gamma} (V^* - \gamma_- Z^*) = K, \tag{30}$$

with

$$\gamma_{\pm} = \tfrac{1}{18}(1+3n) - \tfrac{1}{9}(\delta \pm \Delta)(5+3n), \tag{31}$$

$$\gamma = (\Delta + \Gamma)/(\Delta - \Gamma), \tag{32}$$

$$\Delta = [(\delta - \delta_+)(\delta - \delta_-)]^{\frac{1}{2}}, \quad \Gamma = \delta - (13+3n)/2(5+3n). \tag{33}$$

Near B one has

$$Z^* \zeta^{2+3\gamma_{\pm}} = K'. \tag{34}$$

Since $2+3\gamma_{\pm}$ is positive for $\delta \leqslant \delta_-$ and negative for $\delta \geqslant \delta_+$, in the first case B corresponds to $\zeta = \infty$, and in the second case to $\zeta = 0$. The asymptotic behaviour of h and v near B is given by ($t < 0$)

$$h = -\left(\frac{9}{10+6n}\frac{\nu x^2}{gt}\right)^{\frac{1}{3}}, \tag{35}$$

$$v = \frac{1}{5+3n}xt^{-1}. \tag{36}$$

For $\delta_- < \delta < \delta_0$ and $\delta_0 < \delta < \delta_+$, with $\delta_0 = \frac{13}{10}$ for $n = 0$ and $\delta_0 = 1$ for $n = 1$, B is a focus: the integral curves spiral endlessly towards B counterclockwise as ζ tends to infinity in the first case (stable spiral), and away from B as ζ increases starting from zero at B in the second case (unstable spiral). Near B, when B is a focus, the phase variables V and Z have an oscillatory behaviour that, as is easily verified, the physical variables v and h do not exhibit.

As can be appreciated from table 2, when δ is greater than a critical value δ_c [$\delta_c = 1$ for $n = 0$, and $\delta_c = 0.762\ldots$ for $n = 1$ (as will be shown in §7.7)], the integral curve arriving at the point O originally circles around B. For any n, $\delta_c < \delta_o$; then, in the interval $\delta_c < \delta < \delta_0$, when B represents $\zeta = \infty$ (as the point O does for any $\delta > 0$), the trajectory arriving at O and those arriving at B must originate in some limiting trajectory called a 'limit cycle'. When $\delta_c < \delta < \delta_o$, B is a stable spiral and the existence of at least one limit cycle can be proved by virtue of the Poincaré–Bendixon theorem. Numerical evidence suggests that in this case the limit cycle is unique and that when $\delta > \delta_0$ there is no limiting trajectory (see Lacey *et al.* 1982). For $\delta \rightarrow \delta_0$ the limit cycle tends to B, which for $\delta = \delta_0$ is a centre in the linear approximation. For $\delta \rightarrow \delta_c$ the limit cycle approaches curve $A \rightarrow O$ and the segment OA of the $Z = 0$ axis.

In addition, $Z = Z_B$, $V = V_B$ is also an exact solution of (11), (12) for any n and δ, so that (35), (36) is an exact solution of (2), (3). This special solution (represented in the phase plane by a single point) describes a current with a fixed front at $x = 0$, like (24). When $\delta_- < \delta < \delta_0$, the oscillatory behaviour of the phase variables near B indicates that, as time grows from minus infinity, more and more parcels of the fluid whose flow is described by them, approach the profile of the flow corresponding to the singular point B. A detailed analysis of the solutions that represent fixed fronts will be given later on.

(iv) Point C ($Z_C = 0$, $V_C = \infty$) is a node. Near C the integral curves are given by $Z^{\frac{1}{3}}V = K$. Here C represents a point of the fluid at a finite distance x_f ($= \zeta_f bt^{\delta}$, $\zeta_f = \text{const.}$) from the origin. The following asymptotic formulae are valid near C ($\eta = x/x_f = \zeta/\zeta_f$):

$$Z = (4K)^{\frac{3}{4}}(1-\eta)^{\frac{3}{4}}, \tag{37}$$

$$h = \zeta_f^{\frac{2}{3}}\left(\frac{3b^2\nu}{g}\right)^{\frac{1}{3}}\left(\frac{4K}{3}\right)^{\frac{1}{4}}t^{(2\delta-1)/3}(1-\eta)^{\frac{1}{4}}[1-\tfrac{13}{24}(1-\eta)], \tag{38}$$

$$v = \zeta_f\frac{Kb}{3}\left(\frac{3}{4K}\right)^{\frac{1}{4}}t^{\delta-1}\eta(1-\eta)^{-\frac{1}{4}}. \tag{39}$$

As C is approached ($\eta \rightarrow 1$), $h \rightarrow 0$, $v \rightarrow \infty$ and $(2\pi x)^n hv$ is finite. This behaviour describes the flow near a border of the supporting plate, where the liquid speeds up before falling over, and will be further discussed later on.

(v) Point D ($Z_D = \infty$, $V_D = (1-2\delta)/3(1+n)$) is a saddle. In its neighbourhood the integral curves are given by

$$Z^{-(3+n)/2}(ZV^* - \gamma_0) = K, \quad \gamma_0 = \frac{(2\delta-1)[(5+3n)\delta-1]}{9(3+n)(1+n)^2}. \tag{40}$$

Here $V^* = V - V_D$. Only the curve $V^* = \gamma_0 Z^{-1}$ reaches D from points in the finite (Z, V)-plane. Moving along this curve, D represents the origin $[x = 0]$ and the following asymptotic formulae hold:

$$Z = K'\zeta^{-2}, \tag{41}$$

$$h = (3K'b^2\nu/g)^{\frac{1}{3}}t^{(2\delta-1)/3}, \quad v = -[(2\delta-1)/3(1+n)]\,x/t, \tag{42}$$

which represent gravity spreading with no mass inflow at the origin.

(vi) Point E ($Z_E = \infty$, $V_E = \infty$) is a saddle-node. It represents the origin ($x = 0$). Nearby, (12) can be approximated by

$$\frac{\mathrm{d}Y}{\mathrm{d}W} = \frac{Y(2W+3Y)}{W[(1+n)W-Y]}, \quad Y = Z^{-1}, \quad W = V^{-1}. \tag{43}$$

In the case $n = 0$, integration of (43) yields

$$W^3Y^{-4}(Y+\tfrac{1}{4}W)^5 = K. \tag{44}$$

Besides the solutions $W = 0$ and $Y = 0$, which are of no interest, one obtains from (44) the following asymptotic formulae for the integral curves arriving at E:

$$V = -\tfrac{1}{4}Z \quad (K = 0), \tag{45}$$

and

$$V = \pm 4^{-\frac{5}{8}}K^{-\frac{1}{8}}Z^{\frac{1}{2}} \quad (K \neq 0). \tag{46}$$

From (45) and (46) it can be seen that all the curves that arrive at E in the second and fourth quadrants of the (Z, V)-plane have, for large Z, slopes intermediate between (45) and the $V = 0$ axis. For the curve (45) the following asymptotic formulae are obtained:

$$Z = K'\zeta^{-\frac{5}{4}}, \tag{47}$$

$$h = (3K'b^{\frac{5}{4}}\nu/g)^{\frac{1}{3}}x^{\frac{1}{4}}t^{(5\delta/4-1)/3}, \tag{48}$$

$$v = -(K'b^{\frac{5}{4}}/4)\,x^{-\frac{1}{4}}t^{5\delta/4-1}. \tag{49}$$

This solution represents a current with outflow at the origin. On the other hand the curves (46) lead to

$$Z = K'\zeta^{-2}, \tag{50}$$

$$h = (3K'b^2\nu/g)^{\frac{1}{3}}t^{(2\delta-1)/3}, \tag{51}$$

$$v = \pm 4^{-\frac{5}{8}}K^{-\frac{1}{8}}K'^{\frac{1}{2}}bt^{\delta-1}. \tag{52}$$

These solutions describe currents with inflow or outflow at the origin according to the + or − sign in (52).

In the case $n = 1$, integration of (43) gives

$$YW^3\exp(2W/Y) = K. \tag{53}$$

From (53) it can be seen that the integral curves that reach E are given approximately by

$$V = Z/\ln(Z^2), \tag{54}$$

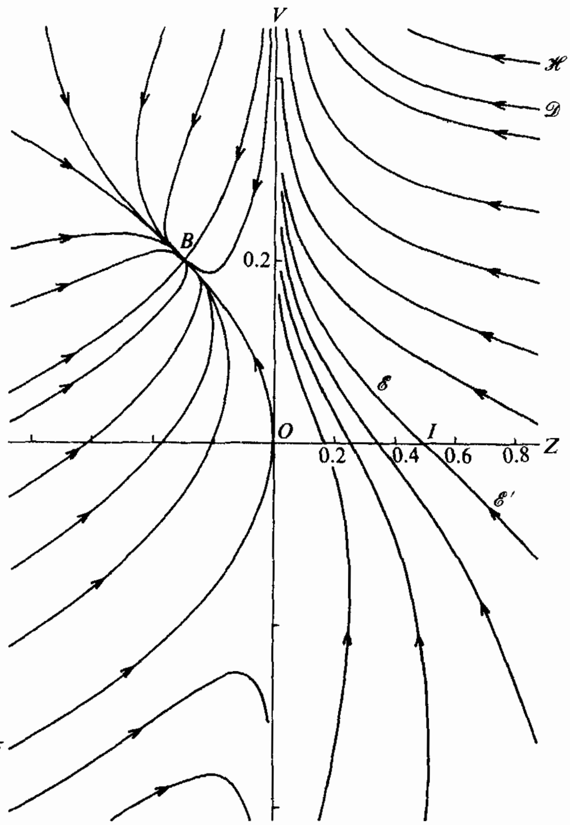

FIGURE 2. The family of integral curves for $n = 0$, $\delta = 0$. Arrows indicate the direction of increasing $|\zeta|$. The curve $\mathscr{F}$ represents a current whose front is fixed. The curves $\mathscr{D}$, $\mathscr{E}$, $\mathscr{E}'$, and $\mathscr{H}$ represent the drainage of a liquid off a supporting plate of finite extent.

and one obtains the following asymptotic formulae:

$$Z = K'\zeta^{-2}|\ln \zeta|^{\frac{3}{4}}, \tag{55}$$

$$h = (3K'b^2\nu/g)^{\frac{1}{3}}t^{(2\delta-1)/3}|\ln \zeta|^{\frac{1}{4}}, \tag{56}$$

$$v = (\tfrac{1}{4}K'b^2)\, x^{-1}t^{2\delta-1}|\ln \zeta|^{-\frac{1}{4}}, \tag{57}$$

describing a current with inflow at the origin. No integral curves in the second and fourth quadrants can arrive at E.

Some families of integral curves are shown in figures 2–5 for the case $n = 0$ and in figures 6, 7 for $n = 1$. The curves have been obtained by numerical integration of (13). In these graphs we have denoted by $\mathscr{D}$ the curve going to the singular point D; the other curves going to infinity arrive either at C or at E.

In table 1 we give a summary of singular points, their properties, the asymptotic behaviour of the solutions, and the corresponding physical interpretation.

A complete list of the various types of trajectories connecting pairs of singular points is given in table 2, where short descriptions of the currents they represent are also given.

It is now convenient to make a brief digression in order to discuss the validity of the solutions that represent currents with sharply defined fronts. Some solutions

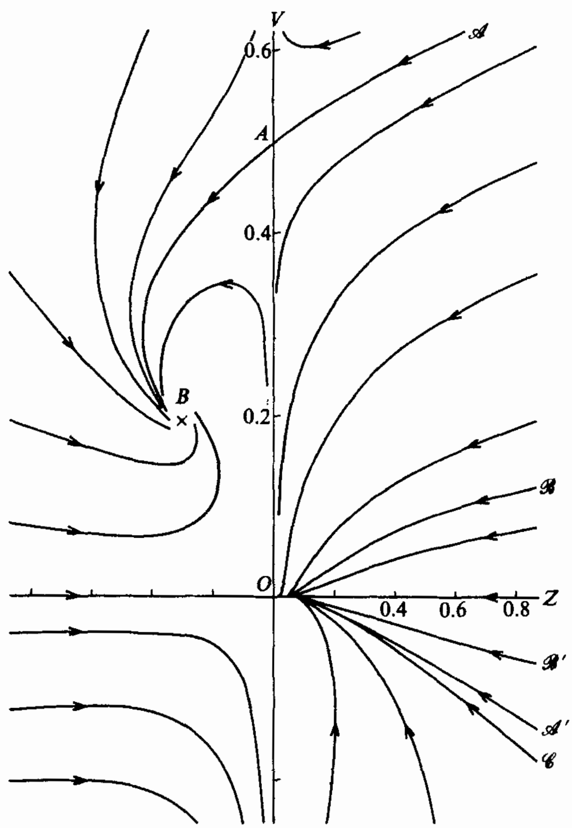

FIGURE 3. The family of integral curves for $n = 0$, $\delta = \frac{1}{2}$. Arrows indicate the direction of increasing $|\xi|$. The curve $\mathscr{A}$ represents the current produced by a source located at $x = 0$. The pair $\mathscr{A}, \mathscr{A}'$ represents the current produced by the removal of a wall that separates a semiinfinite pool of liquid from a initially empty domain. The pair $\mathscr{B}, \mathscr{B}'$ represents the current produced by the removal of a wall that separates two semiinfinite pools of different depth. The curve $\mathscr{C}$ represents the current produced when a wall located at the border of the supporting plate, and containing a pool of liquid, is removed allowing the fluid to fall over.

having this property are those represented by the integral curve $\mathscr{A}$ (equations (25)–(28)) and by the integral curve passing through C (equations (37)–(39)); also the fixed-front solutions (35), (36), as well as others that will be derived in the following sections share this characteristic. In all these cases, the lubrication theory approximation predicts profiles of the form $h \propto X^{\sigma}$, with

$$X = x - x_f \quad \text{and} \quad 0 < \sigma < 1$$

($\sigma = \frac{1}{3}$ for the curve $\mathscr{A}$, $\sigma = \frac{1}{4}$ for the curve through C, $\sigma = \frac{2}{3}$ for the fixed-front solutions). Surely these profiles are incorrect near the front, where the basic assumptions of our model are grossly violated. Nevertheless, by using them it is possible to obtain solutions of the governing equations without invoking further assumptions at the front. Furthermore, the results of the experiments (see Huppert 1982; Didden & Maxworthy 1982; Maxworthy 1983; also Britter 1979) indicate that the theory predicts successfully the overall shape and dynamics of the currents. The matter has been discussed at length in the paper of Huppert; his result is that the conditions at the front of the current play no role in determining its motion or shape

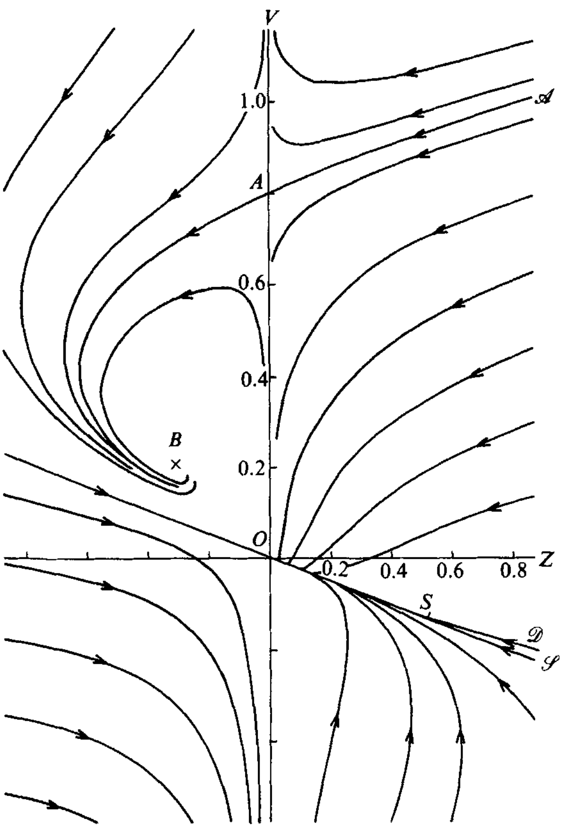

FIGURE 4. The family of integral curves for $n = 0$, $\delta = \frac{4}{5}$. Arrows indicate the direction of increasing $|\zeta|$. The curve $\mathscr{A}$ represents the current produced by a source at the origin. The piece of the curve $\mathscr{S}$ between E and the moving point $S(t)$ represents a steady flow from a constant source at $x = x_s$ to a border of the supporting plate at $x = 0$.

in the large. These investigations were limited to the currents represented by the curve $\mathscr{A}$; however, it certainly looks reasonable on these grounds to expect that also in the other cases the model will describe correctly the general shape and dynamics of the currents, regardless of the fact that the vertical fronts are certainly unrealistic. In general one should expect that wherever the profile predicted by the model becomes appreciably steep it will differ markedly from the real one. In order to estimate the size $\mathscr{X}$ of the region near the front where the theory is certainly incorrect, we may take (somewhat arbitrarily) the condition $(\mathrm{d}h/\mathrm{d}X)_{X=\mathscr{X}} = 1$ as a criterion of steepness. This gives $\mathscr{X} \approx \sigma h$. For profiles that are appreciably steep only very close to the front, such as those we are considering, $\mathscr{X}$ is certainly a very small fraction of the total length of the current, except perhaps at the beginning of the phenomenon.

The case of solutions representing axisymmetric flows from a source at the origin of coordinates (corresponding to the integral curves arriving at E, and whose asymptotic behaviour is given by (55)–(57)) is similar. In this case as $x \rightarrow 0$, h diverges, v tends to zero and the flux $(2\pi xhv)$ tends to a finite value. Here also the model fails where the profile of the current becomes very steep. However, excepting a small region near the origin, the lubrication theory approximation gives predictions in good agreement with the experiments (Huppert 1982).

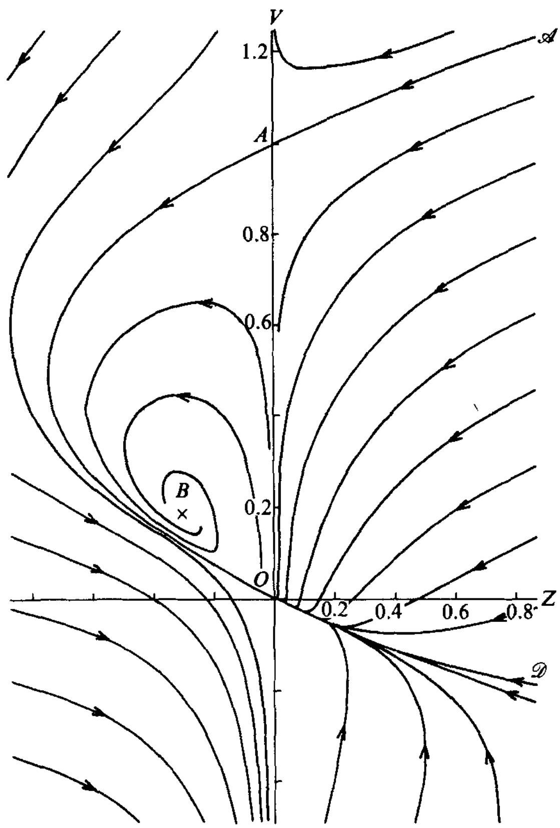

FIGURE 5. The family of integral curves for $n = 0$, $\delta = 1$. Arrows indicate the direction of increasing $|\zeta|$. The curve $\mathscr{A}$ represents the current produced by a source at the origin. It also represents a progressive wave solution, such as the current produced by a piston that is pushing a constant volume of fluid.

4. The special case $n = 0$, $\delta = 0$

This case belongs to plane symmetry and occurs when b has the dimensions of length $[b = \ell,\ \zeta = x/\ell]$. It deserves a separate analysis because, for this choice of n and δ, the general integral of (11) and (12) can be found analytically. To this end let us perform the substitution $Z(\zeta) = W(\zeta)\zeta^{-2}$. One then obtains from (11)

$$V = -\frac{1}{3\zeta}\frac{\mathrm{d}W}{\mathrm{d}\zeta}, \tag{58}$$

and from (12)

$$\frac{\mathrm{d}^2W}{\mathrm{d}\zeta^2} + 1 + \frac{1}{3W}\left(\frac{\mathrm{d}W}{\mathrm{d}\zeta}\right)^2 = 0. \tag{59}$$

A first integral of (59) can be obtained in terms of the function $\Phi(W)$ defined by

$$\frac{\mathrm{d}^2W}{\mathrm{d}\zeta^2} = -\frac{\mathrm{d}\Phi}{\mathrm{d}W}, \tag{60}$$

whose integral is

$$\frac{1}{2}\left(\frac{\mathrm{d}W}{\mathrm{d}\zeta}\right)^2 + \Phi = 0, \tag{61}$$

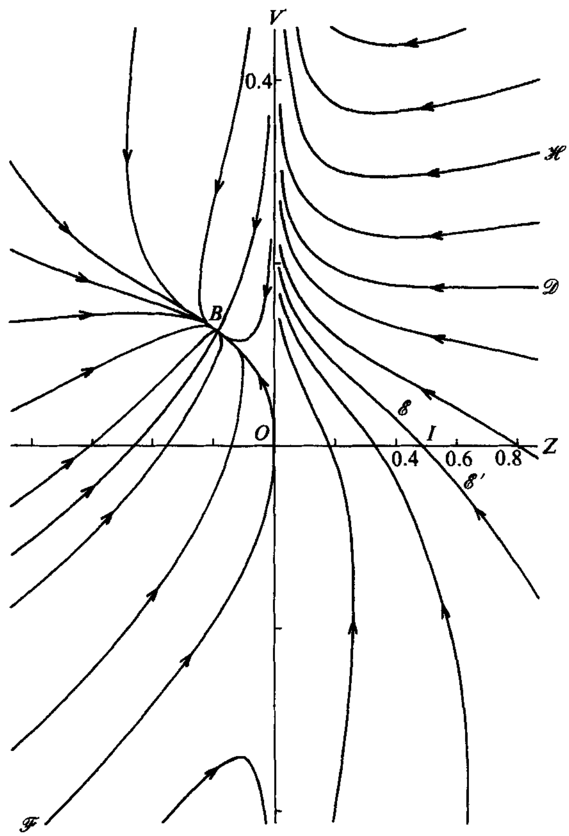

FIGURE 6. The family of integral curves for $n = 1$, $\delta = 0$. Arrows indicate the direction of increasing ζ. The curve $\mathcal{D}$ represents the drainage of a fluid that is allowed to fall over the border of a circular plate. The curve $\mathcal{E}$ represents the drainage off an annular plate, when there is a wall at the inner border. The curve $\mathcal{E}'$ represents the drainage of a circular tank that has a hole at its centre. The curve $\mathcal{H}$ represents a current over a circular plate, produced by a source at the origin. The curve $\mathcal{F}$ represents a current whose front is fixed.

where the integration constant has been set equal to zero. Using (60) and (61), (59) can be written as

$$\frac{\mathrm{d}\Phi}{\mathrm{d}W} = 1 - \frac{2\Phi}{3W}, \tag{62}$$

which yields on integration

$$\Phi = \tfrac{3}{5}W - \tfrac{1}{2}KW^{-\frac{2}{3}}. \tag{63}$$

It can be easily verified that if $K = 0$ one obtains from (63) the analytic solution (24). When $K \neq 0$, introducing (63) in (61) and integrating again we obtain

$$\zeta = \zeta_0 \pm \frac{3}{|K|^{\frac{1}{2}}} (\tfrac{5}{6}K)^{\frac{4}{5}} \int_{w(\zeta_0)}^{w(\zeta)} \frac{|w|^3 \mathrm{d}w}{[\mathrm{sgn}(K)\,(1-w^5)]^{\frac{1}{2}}}, \tag{64}$$

where $w = (6/5K)^{\frac{1}{5}} W^{\frac{1}{3}}$ and $K(1-w^5) \geqslant 0$. The integral in (64) must be computed numerically. Then $w(\zeta)$ can be found by inversion, and one finally obtains the desired solutions in the form

$$h = (\tfrac{5}{6}K)^{\frac{1}{5}} (3\nu\ell^2/gt)^{\frac{1}{3}} w(\zeta), \tag{65}$$

$$v = -\tfrac{1}{3}(\tfrac{5}{6}K)^{-\frac{1}{5}} (\ell/t)\, [K(1-w^5)]^{\frac{1}{2}} w^{-1}. \tag{66}$$

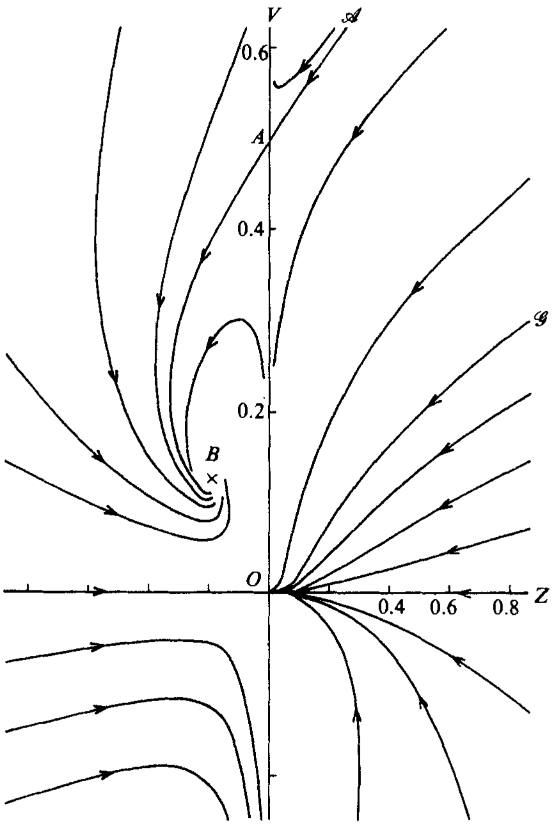

FIGURE 7. The family of integral curves for $n = 1$, $\delta = \frac{1}{2}$. Arrows indicate the direction of increasing ζ. The curve $\mathscr{A}$ represents a current produced by a source at the origin. The curve $\mathscr{G}$ describes the current produced by a constant source at the origin, that is discharging fluid into a preexisting pool.

These solutions describe the drainage of a viscous fluid that is allowed to spill over the borders of the supporting plate, and will be discussed in more detail in §7. It can be observed that $h \propto t^{-\frac{1}{3}}$ and

$$hv = -\tfrac{1}{3}(3\nu\ell^5/gt^4)^{\frac{1}{3}}[K(1-w^5)]^{\frac{1}{2}}. \tag{67}$$

As the position of the border ($w = 0$) is approached, $h \to 0$, $v \to \infty$ but hv is finite.

5. Progressive waves

The basic equations (2) and (3) are translationally invariant in the case $n = 0$, so that they admit solutions of the progressive wave type:

$$h = h(\xi), \quad v = v(\xi), \quad \xi = ct - x, \tag{68}$$

with $c =$ const. In the present case, of course, c does not depend on the properties of the fluid, but is a parameter determined by the boundary conditions, for example a piston moving at a constant speed, and may therefore assume any value.

As is well known there is a close connection between self-similar solutions and progressive values (see for example Barenblatt & Zel'dovich 1972; also Barenblatt 1979). Actually, progressive waves are themselves self-similar, the similarity variable

Point $(Z, V]$	Nature	Interpretation	Behaviour of the physical variables $h \sim$	$v \sim$
$O[0, 0]$	$\delta = 0$: saddle	$x = x_f$: fixed front	$\left[-\frac{(x-x_f)^2}{t}\right]^{\frac{1}{3}}$	$\frac{x-x_f}{t}$
	$\delta \neq 0$: saddle-node	$\delta \neq \frac{1}{2}$: $x = \infty$, mass flow	$x^{\frac{2\delta-1}{3\delta}}$	$x^{\frac{\delta-1}{\delta}}$
		$\delta = \frac{1}{2}$: $x = \infty$, no flow	constant	$\frac{x}{t}\exp(-C\zeta^2)$
$A[0, \delta]$	Saddle	$x \sim t^\delta$, moving front	$t^{\frac{2\delta-1}{3}}\left(1-\frac{x}{x_f}\right)^{\frac{1}{3}}$	$\frac{x}{t}$
$B[\frac{-3}{2(5+3n)}, \frac{1}{5+3n}]$	$0 \leqslant \delta \leqslant \delta_-$: node	$\delta < \delta_0$: $x = \infty$, mass inflow	$\left(-\frac{x^2}{t}\right)^{\frac{1}{3}}$	$\frac{x}{t}$
	$\delta_+ \leqslant \delta$: node	$\delta > \delta$: $x = 0$, fixed point		
	$\delta_- < \delta < \delta_+$: focus			
$C[0, \infty]$	Node	$x \sim t^\delta$, moving sink	$t^{\frac{2\delta-1}{3}}\left(1-\frac{x}{x_f}\right)^{\frac{1}{4}}$	$t^{\delta-1}\frac{x}{x_f}\left(1-\frac{x}{x_f}\right)^{-\frac{1}{4}}$
$D[\infty, \frac{1-2\delta}{3(1+n)}]$	Saddle	$x = 0$, no mass inflow	$t^{\frac{2\delta-1}{3}}$	$-\frac{x}{t}$
$E[\infty, \infty]$	Saddle-node	$n = 0$: $x = 0$, mass inflow or outflow	$x^{\frac{1}{4}}t^{\frac{5\delta-4}{12}}(V \approx \frac{1}{4}Z)$ $t^{\frac{2\delta-1}{3}}$	$x^{-\frac{1}{4}}t^{\frac{5\delta-4}{4}}$ $t^{\delta-1}$
		$n = 1$: $x = 0$, mass inflow only	$t^{\frac{2\delta-1}{3}}\lvert\ln\zeta\rvert^{\frac{1}{4}}$	$\frac{t^{2\delta-1}}{x}\lvert\ln\zeta\rvert^{-\frac{1}{4}}$

TABLE 1. List of the singular points of the phase plane, their position and nature, the corresponding physical interpretation, and the behaviour of the physical variables in their neighbourhood. $\delta_0 = \frac{13}{10}$, $\delta_\pm = \frac{13}{10} \pm (\frac{6}{5})^{\frac{1}{2}}$ for $n = 0$, and $\delta_0 = 1$, $\delta_\pm = 1 \pm \sqrt{3/2}$ for $n = 1$. x_f indicates the position of a front; C denotes a constant. For more details see §3.

being $\lambda = \mu/\tau^c (\mu = \exp(-x/\ell)$, $\tau = \exp(-t/\mathcal{t})$, with ℓ, $\mathcal{t}$ being two governing parameters with dimensions of length and time, respectively, and $\ell = c\mathcal{t}$], and can be obtained from the general formalism of §2 by means of a limiting process, such as that described by Sedov (1959). It is then appropriate to discuss briefly these solutions.

Substituting (68) in (2) and (3) one obtains, denoting with primes the derivatives with respect to ξ

$$v = \mathscr{h}^2\mathscr{h}', \tag{69}$$

and

$$(\mathscr{h}^3\mathscr{h}')' - c\mathscr{h}' = 0, \tag{70}$$

which can be integrated at once giving

$$\mathscr{h}^3\mathscr{h}' - c\mathscr{h} = K. \tag{71}$$

The same result could also have been obtained starting from the formalism of §2 by means of the above-mentioned limiting process.

The solution corresponding to $K = 0$ is

$$h = [(9\nu c/g)(\xi - \xi_0)]^{\frac{1}{3}}, \quad v = c, \quad \xi_0 = \text{const.} \tag{72}$$

Curve	Exists for:	Comments
(*a*) $t < 0$		
$O \to B$	$\delta = 0$	Fixed front [§7.6]
$A \to B$	$0 < \delta < \delta_c$	Converging front with influx at infinity, blows up for $t \to 0$ [§7.7]
$A \to O$	$\delta = \delta_c$	Converging front with influx at infinity. Self-similar solution of the second kind for $n = 1$ [§7.7]
$A \to C$	$\delta > \delta_c$	Converging front with a sink at finite distance, vanishes for $t \to 0$ [§7.7]
$B \to O$	$\delta > \delta_0$	Waiting front with influx from infinity [§7.6]
$B \to C$	$\delta > \delta_0$	Waiting front, sink at finite distance [§7.6]
$LC \to O$	$\delta_c < \delta < \delta_0$	Waiting front with influx from infinity [§7.6]
$LC \to B$	$\delta_c < \delta < \delta_0$	Waiting front with influx from infinity [§7.6]
$LC \to C$	$\delta_c < \delta < \delta_0$	Waiting front, sink at finite distance [§7.6]
$C \to O$	$\frac{1}{2} < \delta < \delta_c$	Influx from infinity to a sink
$C \to B$	$\delta < \delta_c$	Influx from infinity to a sink
$C \to C$	$\delta > \frac{1}{2}$	Current on a plate limited by two sinks
$D \to B$	$\delta < \frac{1}{2}$	Influx from infinity. Current extend to the origin
$D \to C$	$\delta > \frac{1}{2}$	Current from origin to a sink at finite distance
$E \to O$	$\delta < \frac{1}{2}$	Flux from origin to infinity
$E \to B$	only $n = 0$ $\delta \leqslant \frac{1}{2}$	Flux from origin to infinity
$E \to C$	any δ	Flux from origin to a sink at finite distance
(*b*) $t > 0$		
$C \to A$	$\delta < \frac{1}{8}$	Flow towards a sink and expanding over the supporting plate
$D \to A$	$\delta = \frac{1}{5+3n}$	Spreading of a constant volume of fluid [§7.1]
$E \to A$	$\delta \geqslant \frac{1}{8}$ $(n = 0)$ $\delta > \frac{1}{8}$ $(n = 1)$	Current produced by a power-law source at the origin [§§7.1, 7.2, 7.4]
$C \to O$	$\delta \neq 0$	Inflow or outflow at infinity, sink at finite distance
$D \to O$	$\delta > \frac{1}{5+3n}$	Inflow or outflow at infinity, current extends to the origin
$E \to O$	$\delta \geqslant \frac{1}{5+3n}$	Flow from infinity to the origin or vice versa [§§7.2, 7.5]
$C \to C$	$\delta < \frac{1}{8}$	Current on a plate limited by two sinks [§7.3]
$D \to C$	$\delta > \frac{1}{5+3n}$	Current to a sink at finite distance [§7.3]
$E \to C$	any δ	Flux from the origin to a sink at finite distance [§7.3]

TABLE 2. List of the different types of integral trajectories classified according to the singular points they connect. Trivial integral curves are omitted. Column 1 identifies the trajectory. The arrow indicates the direction of increasing $|\zeta|$. *LC* denotes the limit cycle. Each line exists only for the values of intervals of δ indicated in Column 2 ($\delta_0 = \frac{13}{10}$, $\delta_c = 1$ for $n = 0$, and $\delta_0 = 1$, $\delta_c = 0.762\ldots$ for $n = 1$). Column 3 contains short comments on the type of current; the numbers in brackets indicate the paragraph(s) where the corresponding solutions are discussed.

It represents a current that advances with constant speed c on an infinite plate without changing its profile, and whose front is located at $x = ct - \xi_0$. This current describes the flow produced by a plane piston (or a spatula) that is advancing at a constant speed, pushing a constant volume of fluid in front of it. It can be easily checked that this solution can also be derived from the formalism of §3, and that

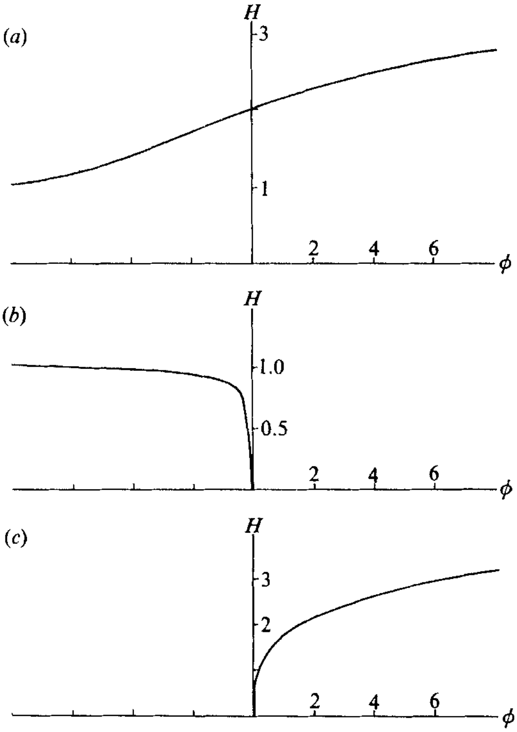

FIGURE 8. Profiles of the progressive wave solutions: (*a*) the current produced by a piston pushing a layer of fluid of constant depth. (*b*) The current produced when a liquid is falling over a border that is receding at a constant speed. (*c*) The current produced when a liquid is pushed towards a border that is advancing at a constant speed.

it belongs to the $n = 0$, $\delta = 1$ family, being described by the integral curve $\mathscr{A}[Z = 3V(V-1)]$.

Analytic solutions can also be obtained from (71) in the case $K \neq 0$. Let us consider first $K < 0$. We put

$$K = -c\hbar_0, \quad \hbar = H(\phi)\hbar_0, \quad \phi = c\xi/\hbar_0^3, \tag{73}$$

then (71) is transformed into

$$\frac{d\phi}{dH} = \frac{H^3}{H-1}. \tag{74}$$

Two cases can be considered: (*a*) $H \geqslant 1$, (*b*) $0 \leqslant H \leqslant 1$. In case (*a*) integration of (74) yields

$$\phi = \tfrac{1}{3}(H-1)^3 + \tfrac{3}{2}(H-1)^2 + 3(H-1) + \ln(H-1) - \tfrac{29}{6}, \tag{75}$$

where the integration constant has been chosen such that $H(\phi = 0) = 2$. The profile given by (75) is shown in figure 8(*a*). It represents a gravity current whose profile does not change with time and that advances with constant velocity c; the thickness of the fluid tends to $h_0 = (3\nu/g)^{\frac{1}{3}}\hbar_0$ at great distances in front of the current and increases as $|x|^{\frac{1}{3}}$ far behind. There is no sharply defined front, but the point given by $\phi = 0$ [where $h = 2h_0$] can be conventionally taken as the position of the 'front'. The average velocity of the fluid in this current tends to zero ahead, and approaches the constant value c far behind the advancing 'front'. This solution describes the asymptotics of the current produced by a piston that is pushing a layer of fluid of thickness h_0, when a sufficiently long time has elapsed from the beginning of the motion, and the front of the perturbation has advanced to a very great distance from the piston.

In case (*b*) one obtains

$$\phi = -\tfrac{1}{3}(1-H)^3+\tfrac{3}{2}(1-H)^2-3(1-H)+\ln(1-H)+\tfrac{11}{6}, \tag{76}$$

where the integration constant has been chosen such that $H(\phi = 0) = 0$. The profile (76) is represented in figure 8(*b*). This current has a well-defined front at $\phi = 0$; behind this front (to the right, in the figure) there is no fluid at all. Ahead of the front, the thickness of the fluid layer tends rapidly to h_0. The average flow velocity is opposite to that of the profile ($= c$), it vanishes rapidly as $h \rightarrow h_0$, and tends to infinity as the front is approached. Near the front one has, approximately,

$$\phi = -[\tfrac{1}{4}H^4+\tfrac{1}{5}H^5+\ldots]. \tag{77}$$

This solution represents a viscous liquid flowing towards, and over a moving border of the supporting plate. Such a current occurs if a fluid layer of uniform thickness h_0 is initially at rest, and at a certain moment the supporting plate begins to be gradually destroyed, so that its border recedes at a constant speed, letting the fluid spill over it.

Let us consider the case $K > 0$. We change the definition of h_0 setting $K = ch_0$ in (71) and obtain

$$\frac{\mathrm{d}\phi}{\mathrm{d}H} = \frac{H^3}{H+1}. \tag{78}$$

Integration of (78) gives

$$\phi = \tfrac{1}{3}(1+H)^3-\tfrac{3}{2}(1+H)^2+3(1+H)-\ln(1+H)-\tfrac{11}{6}, \tag{79}$$

having chosen the integration constant such that $H(\phi = 0) = 0$. This profile is shown in figure 8(*c*), and corresponds to quite far-fetched boundary conditions: like the preceding case, the current is flowing towards a border of the supporting plate but now the border is moving away from the fluid, while the fluid is being pushed towards it by an advancing piston (located very far from the front of the current). Both the border and the piston move with the same velocity c.

In contrast to (72), the solutions (75), (76), and (79) are not represented in the phase plane. In general, the progressive waves do not admit a self-similar representation of the type (6), (10), as they depend on two dimensional parameters, c and h_0, with independent dimensions. When $h_0 = 0$ (and then $K = 0$) the governing parameters are reduced to one, and self-similarity of the type (6), (10) is again possible.

6. Steady flows

It is easy to show that the basic equations (2), (3) admit the time-independent solutions

$$h = \left[\left(\frac{4q\nu}{g}\right)\left(x_0-x\right)\right]^{\frac{1}{4}}, \quad v = q\left[\left(\frac{4q\nu}{g}\right)\left(x_0-x\right)\right]^{-\frac{1}{4}}, \quad hv = q, \tag{80}$$

with $q = \text{const.}$, $x_0 = \text{const.}$, for $n = 0$, and

$$h = \left[\left(\frac{4q\nu}{\pi g}\right)\ln\left(\frac{x_0}{x}\right)\right]^{\frac{1}{4}}, \quad v = \left(\frac{q}{4\pi}\right)^{\frac{3}{4}} x^{-1}\left[\left(\frac{\nu}{g}\right)\ln\left(\frac{x_0}{x}\right)\right]^{-\frac{1}{4}}, \quad 2\pi xvh = q, \tag{81}$$

for $n = 1$. These solutions describe a steady flow on a finite plate, from a constant source at $x = 0$ to a border at $x = x_0$.

It can be easily verified that (80) is also a self-similar solution of the $n = 0$, $\delta = \frac{4}{5}$ family, represented by a piece of the integral curve $\mathscr{S}$ (see figure 4) given by $V = -\frac{1}{4}Z$, which is an exact solution of (13). On the other hand, (81) does not admit a self-similar representation of the type (6), (10).

The connection between the steady flows and the self-similar currents can be clarified observing that for a stationary solution the representation (6) is no longer valid and must be replaced by

$$h = [x^2 Z(\zeta)/t]^{\frac{1}{3}}, \quad v = xV(\zeta)/t, \tag{82}$$

where t is a constant governing parameter whose dimensions are T, in addition to b, which must have in this case the dimensions of length. One then obtains:

$$\zeta Z' + 2Z + 3V = 0, \tag{83}$$

$$3\zeta Z V' + \zeta Z' V + (5+3n)\, VZ = 0, \tag{84}$$

instead of (11) and (12). Then a steady flow will not be self-similar, unless the condition $\delta \zeta Z' + Z = 0$ is satisfied. Except for trivial solutions this happens only in the special case $n = 0$ and $\delta = \frac{4}{5}$, and is not possible if $n = 1$.

We observe that the steady flows, like the progressive waves, are a special case of a family of solutions called limiting to self-similar by Barenblatt (1979), and that they can be derived from the general formalism (§2) by means of an appropriate limiting process.

7. Construction of solutions for specific problems

In this Section we shall indicate briefly how to derive from the formalism the solutions for some specific problems. A more detailed analysis of the properties of these solutions will be given in a forthcoming paper.

Two steps are usually needed to find the integral curve (or curves) that represents the solution of a given problem: first, one must identify by inspection of the boundary (or initial) conditions the parameter b that determines the self-similar variable ζ and the exponent δ; second, one has to select the appropriate integral curve among the (n, δ) family (i.e. the curve whose asymptotic behaviour corresponds to the boundary conditions at hand). The solutions thus found are called 'self-similar of the first kind' (Barenblatt 1979). In certain instances, the boundary or initial conditions do not determine the parameter b, so that the self-similarity exponent δ must be found by other methods, as we shall show in §7.7. The solutions are then called 'self-similar of the second kind' (Barenblatt 1979). Let us discuss a few examples. The reader can find in tables 1 and 2 a summary of the properties of the singular points and of the integral curves relevant for the flows discussed in this section.

7.1. *Viscous gravity currents whose volume varies with time according to a power law*

This is the problem studied by Huppert (1982). These flows obey the global continuity equation:

$$\int_0^{x_f(t)} (2\pi x)^n\, h(x,t)\, \mathrm{d}x = q_\alpha t^\alpha, \tag{85}$$

with $q_\alpha = $ const.; thus $\alpha = 0$ corresponds to the spread of a constant volume of fluid, $\alpha = 1$ to a source of constant flux at $x = 0$, etc. The supporting plate extends to infinity. As stated in §3 these flows are represented by the segment of the curve $\mathscr{A}$ (see figures 3–5, 7) that joins the singular points A and D (for $\alpha = 0$), or E (for $\alpha \neq 0$). Using (4), (6), and (10) in (85) one obtains

$$\delta = \frac{1+3\alpha}{5+3n}, \quad b = [gq_\alpha^3/3\nu(2\pi)^{3n}]^{1/(5+3n)}, \tag{86}$$

and

$$\zeta_f = \left[\int_0^1 (\eta^{2+3n} Z)^{\frac{1}{3}} \mathrm{d}\eta\right]^{-3/(5+3n)}. \tag{87}$$

All the results of Pattle (1959), Smith (1969), Nakaya (1974), Lopez *et al.* (1976), and Hupert (1982) can be recovered by means of the phase-plane formalism. In addition, we observe that in the case $n = 0$, $\alpha = \frac{4}{3}[\delta = 1]$ the solution [corresponding to the curve (29)] is analytic, a fact not previously noticed. It is given by

$$h = 6^{\frac{1}{3}}[2q_\alpha^3 \nu^4/g^4]^{\frac{1}{15}} (ct-x)^{\frac{1}{3}}, \quad v = c = \tfrac{2}{3}(2gq_\alpha^3/\nu)^{\frac{1}{5}}. \tag{88}$$

It can be recognized that this solution is partially coincident with the progressive wave solution (72).

7.2. *Viscous gravity currents produced by the removal of a wall that separates two layers of fluid of different thickness*

This problem is analogous to that of the breaking of a dam for an inviscid fluid (see for example Whitham 1974): a thin vertical wall located at $x = x_0$ separates a layer of fluid whose thickness is h_0 in the $x < x_0$ region from a layer of thickness h_0' in the region $x > x_0$; the supporting plate extends to infinity; at $t = 0$ the wall is suddenly removed and a gravity flow ensues tending to restore equilibrium.

The governing parameters of this problem are $\hbar_0 = (g/3\nu)^{\frac{1}{3}} h_0$ and $\hbar_0' = (g/3\nu)^{\frac{1}{3}} h_0'$, whose dimensions are $(L^2/T)^{\frac{1}{3}}$, and x_0 which has the dimensions of L. Clearly the flow will not be self-similar in the axisymmetric case ($n = 1$). For plane symmetry ($n = 0$), translational invariance allows the choice of $x_0 = 0$ so that in this case there is a self-similar solution. The similarity variable can be taken as

$$\zeta = x/\hbar_0^{\frac{3}{2}} t^{\frac{1}{2}}, \tag{89}$$

i.e. $b = \hbar_0^{\frac{3}{2}}$, $\delta = \frac{1}{2}$. The solution depends on $\hbar_0$ and on the dimensionless parameter $\pi = \hbar_0'/\hbar_0$. The phase plane is represented in figure 3. Let us assume $\hbar_0 > \hbar_0'$ for definiteness. The solution is represented by two pieces: a curve such as $\mathscr{B}$ joining E and O, that represents the flow in the domain $x > 0\,(= x_0)$, and a curve such as $\mathscr{B}'$ going from E to O in the $V < 0$ half-plane that represents the flow in the domain $x < 0$. For $x \to -\infty$ and $x \to +\infty$, h must approach h_0 and h_0' respectively, and the solutions corresponding to $\mathscr{B}$ and $\mathscr{B}'$ must be joined smoothly at $x = 0$ by requiring the continuity of h and v. These conditions determine uniquely the pair $\mathscr{B}$, $\mathscr{B}'$ and the integration constants.

If $h_0' = 0$, i.e. no fluid is initially present in the $x > 0$ domain, the solution must have an advancing front at a finite distance from the origin. The appropriate integral curves are $\mathscr{A}$, going from E to A, and $\mathscr{A}'$ (see figure 3), where the latter is determined by the requirements of the continuity of h and v as above. It can be observed that the current in the $x > 0$ domain is identical to that produced by a source at the origin whose flux varies as $t^{-\frac{1}{2}}$, that is, a flow of the type described in §7.1, with $\alpha = \frac{1}{2}$, and $q_\alpha = (gh_0^5/3\nu)^{\frac{1}{2}}$.

Now suppose that the supporting plate extends only from $x = -\infty$ to $x = 0$. Then when the wall is removed the fluid begins to fall over the plate border. This current is represented by the curve $\mathscr{C}$ that joins E and O, and that near E is given by the approximate formula (45).

We observe that these flows are characterized by a vertical scale h_0 constant in time, so that the only change of the profile of the current as time elapses is an horizontal stretching proportional to $t^{\frac{1}{2}}$.

7.3. *Drainage of a viscous fluid from a finite supporting plate*

Let us assume that a certain quantity of fluid is initially contained between two vertical walls located at $x = x_0$ and $x = x_0'$, respectively, and that the supporting plate does not extend beyond the walls. At $t = 0$ one of the walls (that at x_0, say), or both, are suddenly removed, so that the fluid falls over the plate border, or borders. We are interested only in the last stages of the flow, when most of the fluid has already drained and the average depth of the fluid is much less than the initial one. Clearly, in this case the initial depth of the fluid is not relevant so that the only governing parameters are x_0 and x_0', whose dimensions are L. The flow is then self-similar (both for the plane and the axisymmetric case) and $\delta = 0$. This is a very special case of self-similarity, as ζ depends only on x, so that h and v depend on time only through the $(x^2/t)^{\frac{1}{3}}$ and (x/t) scales of (6). Let us discuss the axisymmetric case ($n = 1$); the phase plane is shown in figure 6. The flows we are considering are represented by integral curves such as $\mathscr{D}$, $\mathscr{E}$, $\mathscr{E}'$ and $\mathscr{H}$. The curve $\mathscr{D}$, joining D and C, represents the drainage of a fluid supported by a circular plate (x_0 = radius of the plate, $x_0' = 0$). The curve $\mathscr{E}$, from the point $I[Z = Z_I, V = 0]$ to C represents the drainage of an annular plate when the external wall (at $x = x_0$) is removed but the inner wall (at $x = x_0'$) is retained. We mention here that, when $\delta = 0$, the portion of any trajectory ending on the line $V = 0$, as in the present case, represents a flow with a wall at the corresponding point (x_0). The curve $\mathscr{E}'$, from C to I, describes the drainage of a circular tank whose radius is x_0' through a hole of radius x_0 at its centre. Taking the curves $\mathscr{E}$ and $\mathscr{E}'$ together as a single piece one can construct the solution corresponding to the drainage of an annular plate when the fluid is allowed to spill over both the inner and the outer border. Finally, the curve $\mathscr{H}$ describes the current due to a source at the origin, whose flux varies with time as $t^{-\frac{4}{3}}$, over a circular plate of radius x_0. Analogous solutions for the $n = 0$ case can be found in a similar way.

7.4. *Dipole-type solutions*

By analogy with the case of constant volume, obtained as a particular solution in §7.1, one can seek solutions which preserve certain integral moments of h during the spreading of the flow:

$$M = \int_0^{x_f(t)} (2\pi x)^n \, x^\nu h(x,t)\,\mathrm{d}x = \text{const.}, \tag{90}$$

where ν is an arbitrary constant. Using (4), (6), and (10) in (90) one finds

$$\delta = 1/(5+3n+3\nu), \tag{91}$$

and

$$b = \left(\frac{g}{3\nu}\right)^{\delta}\left[\frac{M}{(2\pi)^n}\right]^{3\delta}, \quad \zeta_f = \left[\int_0^1 \eta^{\frac{1}{3\delta}-1} Z^{1/3}\,\mathrm{d}\eta\right]. \tag{92}$$

We mention in passing that condition (90) is completely equivalent to (85), the exponents α and ν being related by

$$\alpha = -\nu/(5+3n+3\nu) = -\delta/\nu. \tag{93}$$

For plane symmetry, the condition of constancy of the first moment ($\nu = 1$), requires $\delta = \frac{1}{8}$ and, according to (92), corresponds to a sink in the origin whose flux varies as $t^{-\frac{9}{8}}$ while a front moves away. For this value of δ the integral curve leaving A (which represents the moving front), must join E (which represents a sink at the origin) (see tables 1 and 2). This trajectory is analytic and is given by

$$V = \tfrac{1}{8} - \tfrac{1}{4}Z. \tag{94}$$

In terms of the physical variables the solution is

$$h = (\tfrac{3}{10} b^2 \zeta_f^2)^{\frac{1}{3}} t^{-\frac{1}{4}} \eta^{\frac{2}{3}} (\eta^{-\frac{5}{4}} - 1)^{\frac{1}{3}}, \tag{95}$$

$$v = \tfrac{1}{5} b \zeta_f t^{-\frac{7}{8}} \eta (1 - \tfrac{3}{8} \eta^{-\frac{5}{4}}), \tag{96}$$

where $\eta = \zeta/\zeta_f$, and b and ζ_f are constants. We observe that as the integral curve (93) crosses the $V = 0$ axis, there is a (moving) point in the current (at $\eta = \eta_0 = (\frac{3}{8})^{\frac{4}{5}}$) that separates the region in which the liquid flows towards the sink ($0 < \eta < \eta_0$) from that in which the liquid flows towards front ($\eta_0 < \eta < 1$). The height h of the current is a maximum at $\eta = \eta_0$. As this solution conserves the first-order moment (90), it can be called of the dipole type, in analogy with the solutions obtained by Barenblatt (1954) and Barenblatt & Zel'Dovich (1957) in the context of the porous media equation, see also Zel'dovich & Raizer (1967) for solutions of this type in relation with nonlinear heat diffusion.

7.5. *Currents produced by a source at the origin*

This is a problem of the same type as that discussed in §7.1, only slightly more general. Let us assume that the flux $f = (3\nu/g)^{\frac{1}{3}} \mathscr{f}$ of the source depends on time according to the power law

$$\mathscr{f} = \lim_{x \to 0} (2\pi x)^n \mathscr{h} v = Q_\beta t^\beta, \tag{97}$$

where β is some constant. The dimensions of the governing parameter Q are

$$[Q] = L^{(5+3n)/3} T^{-(4+3\beta)/3}, \tag{98}$$

so that

$$\delta = \frac{4+3\beta}{5+3n}. \tag{99}$$

If the boundary conditions do not introduce other dimensional parameters in addition to Q, the problem is identical with that of §7.1 for $\alpha = \beta + 1$ and $q_\alpha = (3\nu/g)^{\frac{1}{3}} (\beta+1)^{-1} Q$. But if an additional governing parameter enters into the problem other solutions can be found. In general, these will not be self-similar. However, when the dimensions of the additional parameter and of Q are mutually dependent, as may happen for some special values of β, one still obtains self-similarity. Let us consider some interesting examples.

It is easy to check that when $\beta = \frac{1}{2}(n-1)$ the dimensions of $\mathscr{h}$ and of Q are dependent so that problems involving an additional parameter $\mathscr{h}_0$ are self-similar. Cases of this type occur if one considers the flow from a source into a pool of liquid having initially a uniform depth h_0.

Let us consider first the axisymmetric case. Then $\beta = 0$, i.e. the source has a constant flux, and $\delta = \frac{1}{2}$. The current is represented in the phase plane by the curve $\mathscr{G}$ joining E and O (see figure 7), which corresponds to a solution having the correct asymptotic properties as can be easily checked.

The analogous solution in the plane case corresponds to $\beta = -\frac{1}{2}$ and $\delta = \frac{1}{2}$. It is represented by the integral curve $\mathscr{B}$ (figure 3) already discussed in §7.2 in connection with the current produced by the removal of a wall separating two pools of liquid having different depths.

Other examples of self-similar flows obtained when the dimensions of two or more governing parameters are mutually dependent can be devised, but will not be discussed here for the sake of brevity.

7.6. *Viscous gravity currents having a fixed front*

Various solutions of this type appear in the formalism, as mentioned in §3, where we remarked that they are meaningful for $t < 0$. Summarizing, one has:

(*a*) The flow given by (35) and (36), which are exact solutions of the governing equations (2) and (3). This flow is represented in the phase plane by the point B (that can be regarded as a singular integral curve), irrespective of the values of n and δ, which do not appear in the solutions (35), (36). It can be observed that for this particular flow h and v do not depend on any dimensional parameter (besides ν/g, only numerical constants enter in (35), (36)). The profile of this flow varies as $x^{\frac{2}{3}}$ and the thickness at a fixed position increases infinitely with time as $(-t)^{-\frac{1}{3}}$ for $-t \rightarrow 0$. The flow velocity is zero at the front at any fixed time and increases linearly with x, while for fixed x, it increases as $(-t)^{-1}$, being directed towards the front. The equation of motion of a parcel of the fluid that is moving with the average velocity is given by

$$x = K(-t)^{1/(5+3n)}, \tag{100}$$

so that at $t = 0$ all such parcels collapse at the front, where the thickness becomes infinite. This behaviour corresponds to the so-called 'waiting time' solutions that all nonlinear diffusion equations like (5) possess (see Lacey *et al.* 1982). Physically the fluid has a front that, under appropriate conditions and without any wall stopping it, waits a finite amount of time for the rest of the fluid to reaccommodate before spreading.

(*b*) For $\delta = 0$ the integral curve $\mathscr{F}$ (see figures 2 and 6) arrives at O. It is given for $n = 0$ by (24) which is an exact solution of (13). The corresponding flow coincides with that discussed in (*a*), except for a translation. For $n = 1$, (24) represents the asymptotic behaviour near O of $\mathscr{F}$, there being no analytic solution of (13) in this case. The asymptotic behaviour of h and v is

$$h = \left[-\frac{9\nu}{10gt}(x-x_0)^2\right]^{\frac{1}{3}}, \quad v = \frac{1}{5t}(x-x_0), \tag{101}$$

and represents a current with a fixed front at $x = x_0$.

(*c*) For any n, and $\delta > \delta_0$, the singular point B represents the origin of coordinates. For the integral curves leaving B, the asymptotic behaviour of h and v near the origin is given by (35) and (36). As in the case (*a*) above, these solutions represent currents with stationary fronts at the origin. The difference between the present case and (*a*) is that now the boundary conditions elsewhere involve a dimensional parameter b so that δ is no longer undetermined (in addition these boundary conditions single out which of the infinite integral curves leaving B must be chosen). The boundary conditions influence the flow far from the origin, so that the stationarity of the front is not sufficient to determine the behaviour of the current at large distances.

(*d*) As was mentioned at the end of §3, when $\delta_c < \delta < \delta_0$ there is a limit cycle surrounding B. As the limit cycle is approached ($\zeta \rightarrow 0$) the phase variables Z and V become oscillatory functions of ζ although it can be easily proved that the physical variables h, v are monotonic functions of x for any t. Then, according to (6), the limit cycle itself represents a stationary front and trajectories emerging from it represent waiting-type solutions.

From that discussed in points (*c*) and (*d*) above, solutions of the waiting type can be constructed for any δ greater than δ_c. These kinds of solutions have been extensively studied in the literature in the context of nonlinear diffusion equations (see, for instance, Lacey *et al.* 1982; Kath & Cohen 1982; Smyth & Hill 1988), and represent by themselves a vast area of research. The problem of finding the appropriate continuation of the waiting time solutions for $t > 0$, when the front begins to move, has been studied by Lacey *et al.* (1982). For the sake of brevity we shall not discuss these solutions further, and refer the interested reader to the above-mentioned literature.

7.7. *Collapse of an axisymmetric converging current: a case of self-similarity of the second kind*

Let us imagine the axisymmetric flow of a viscous fluid towards the origin as might occur, for example, if we have initially a pool of fluid outside a circular wall, while inside the wall there is no liquid, and the wall is suddenly removed, letting the liquid run towards the centre. The flow will have a convergent front, whose radius reduces as the fluid spreads, and that will finally collapse at the origin. Let us consider the last stages of the process, near the collapse of the front. We shall be interested in the properties of the flow for small radii, compared with any constant parameter characteristic of the initial conditions, say, for instance, the radius of the circular wall containing initially the fluid. In this situation we are left with no constant governing characteristic parameters: those arising from the initial conditions are no longer suitable as a scale of the region of interest, and the characteristic parameters of the flow therein are functions of time. Consequently the flow will still be self-similar, but the similarity exponent δ cannot be determined by dimensional considerations. This is then a case of self-similarity of the second kind (see Barenblatt & Zel'dovich 1972 and also Barenblatt 1979).

As the current has an advancing front, the solution must be represented by a trajectory leaving the singular point A. Also, we are interested in what happens before the front collapses, i.e. for $t < 0$, the collapse corresponding to $t = 0$. This means that the integral curve of interest must lie in the $Z < 0$ half-plane. For $Z < 0$, the single trajectory leaving A can go either to B, O or C; a curve joining A with B (or A with C) represents a flow that blows up (or fades away) as t approaches 0 at points at a finite distance from the origin; then, it cannot represent the solution we are looking for. The actual flow must then be represented by a trajectory, or a portion of it, joining A and O, which has the required property of describing a flow with h, v finite and non-zero for $t \rightarrow 0$ at any finite distance behind the front. Such a curve exists only for a particular value of δ, which we shall call δ_c (see figure 9). By numerical evaluation one finds $\delta_c = 0.762\ldots$. The solution is then given, near the front, by

$$h = \left(-\frac{9\nu}{g}\delta_c\frac{x_f^2}{t}\right)^{\frac{1}{3}}(1-\eta)^{\frac{1}{3}}\left[1-\frac{4\delta_c-1}{24\delta_c}(1-\eta)+\ldots\right], \tag{102}$$

$$v = -\frac{\delta_c x_f}{t}\eta\left[1+\frac{8\delta_c-1}{4\delta_c}(1-\eta)+\ldots\right], \tag{103}$$

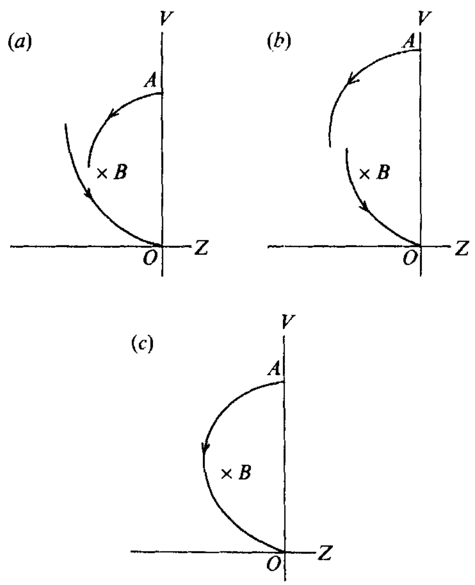

FIGURE 9. The eigenvalue problem for the axially converging front current. For $t < 0$, A and O are saddle points. An integral curve joining A with O can be found only for $\delta = \delta_c$. (a) $\delta < \delta_c$; (b) $\delta > \delta_c$; (c) $\delta = \delta_c$.

where $\eta = x/x_f$, and $x_f = K(-t)^{0.762\cdots}$. It can be seen that as $t \rightarrow 0$, the front speeds up, its velocity tending to infinity at the instant of the collapse.

Thus we have seen that in the present case the self-similarity exponent δ_c is determined by solving a nonlinear eigenvalue problem (and not by dimensional analysis alone, as is the case of self-similar solutions of the first kind). This is typical of self-similar problems of the second kind.

It can also be noticed that the present problem is strongly reminiscent of that of the collapse of cylindrical and spherical shock waves in gas dynamics that leads to the classical solutions of Guderley (1942), see for example Zel'dovich & Raizer (1967). The complete discussion of this type of solution and the analogous ones that can be expected in the related problems of nonlinear diffusion, nonlinear heat conduction, etc. are left for future work.

8. Summary and conclusions

The phase-plane formalism we have developed for viscous gravity flows described by the lubrication theory approximation is based on the analogous formalisms of Sedov (1959) and of Courant & Friedrichs (1948) for gas dynamics. It allows the systematic derivation of the similarity solutions represented by scales that depend on the spatial and temporal variables according to power laws of the type (6) and (10). The solutions are represented by integral curves in the plane of the phase variables Z and V, which are related to the depth and the average horizontal velocity of the fluid. Each integral curve corresponds to a certain self-similar gravity current satisfying a particular set of initial and boundary conditions.

All conceivable boundary and initial conditions (compatible with self-similarity of the type considered) are represented in the phase plane, so that the present theory is complete in the sense that it contains all the self-similar currents described by the governing equations. A detailed analysis of the properties of the integral curves in the neighbourhood of the singular points of the phase plane has been carried out, and the asymptotic formulae describing the behaviour of the physical quantities are given. This allows the determination of the appropriate solution for specific problems. We have illustrated by means of various examples how to derive from the formalism the desired self-similar solutions. The examples discussed include, in addition to the similarity flows studied by other authors, novel self-similar solutions of the first kind such as the extension to viscous flows of the classic problem of the breaking of a dam, flows on finite plates, and several others, and self-similar flows of the second kind, such as the collapse of an axisymmetric converging current. Some interesting analytic solutions have been found in various cases.

An investigation of solutions of other types, such as progressive waves and steady flows has been included owing to their close connection to self-similar flows.

With slight modifications, the phase-plane formalism of the present paper can be employed to investigate the analogous self-similar solutions for other physical processes governed by nonlinear parabolic equations, such as nonlinear heat conduction, nonlinear diffusion, motion of ground water, etc.

The detailed study of some of the solutions we have found, including in certain cases the analysis of their stability, is left for a forthcoming paper.

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